3.7

$$\langle x; y\rangle = y^T \underbrace{\begin{pmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{pmatrix}}_{G} x \quad *$$

$$S_1 = \{ x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \} = gen \{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{v_1} \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{v_2} \}$$

$$S_1^\perp = \{ v \in \mathbb{R}^3 : v \perp v_1 , v \perp v_2 \}$$ ojo con pi *

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \langle v; v_1\rangle = 0 \qquad (1 \quad -1 \quad 0)\, G \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4x_1 + (-7)x_2 + (-4x_3) = 0 \quad (1)$$

$$\langle v; v_2\rangle = 0 \qquad (0 \quad 1 \quad -1)\, G \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2x_1 + x_2 - 2x_3 = 0 \quad (2)$$

$$\begin{array}{c} (1) \\ (2) \end{array} \begin{pmatrix} 4 & -7 & -4 \\ -2 & 1 & -2 \end{pmatrix} 2F_2 + F_1 \qquad \begin{pmatrix} 4 & -7 & -4 \\ 0 & -5 & -8 \end{pmatrix} \qquad -5x_2 = +8x_3 \qquad \boxed{x_2 = -8/5\, x_3}$$

$$4x_1 - 7\left(-8/5\, x_3\right) - 4x_3 = 0$$

$$4x_1 = 4x_3 - 56/5\, x_3 = -36/5\, x_3 \qquad x_1 = -9/5\, x_3$$

$$S_1^\perp = gen\{ \underbrace{\begin{pmatrix} -9 \\ -8 \\ 5 \end{pmatrix}}_{u} \}$$

$$\underset{S_1^\perp}{P}(v) = \frac{\langle v; u\rangle u}{\| u \|^2} \; *$$

$$\underset{S_1^\perp}{P}(v) = \frac{\langle v, u \rangle \, u}{\| u \|^2} \qquad v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\frac{\begin{pmatrix} -9 & -8 & 5 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} -9 \\ -8 \\ 5 \end{pmatrix}}{} =$$

$$\frac{-2x_1 - 2x_2 - 2x_3}{24} \begin{pmatrix} -9 \\ -8 \\ 5 \end{pmatrix} \qquad \left[\underset{S_1^\perp}{P}(v)\right]^{E} = -\frac{1}{12} \begin{pmatrix} -9 & -9 & -9 \\ -8 & -8 & -8 \\ 5 & 5 & 5 \end{pmatrix}$$

$$\|u\|^2 = \begin{pmatrix} -9 & -8 & 5 \end{pmatrix} \begin{pmatrix} 2 & -2 & 0 \\ -2 & 5 & 4 \\ 0 & 4 & 6 \end{pmatrix} \begin{pmatrix} -9 \\ -8 \\ 5 \end{pmatrix} = -2(-9) - 2(-8) - 2.5$$

$$= 18 + 16 - 10 = 24$$

2. Sea $T : \mathbb{R}^3 \to \mathbb{R}_2[x]$ la transformación lineal definida por

$$[T]^{\mathcal{C}}_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

donde $\mathcal{B}$ y $\mathcal{C}$ son las bases de $\mathbb{R}^3$ y $\mathbb{R}_2[x]$, respectivamente, definidas por

$$\mathcal{B} = \{ \overset{v_1}{[6 \ \ 3 \ \ 2]^T}, \overset{v_2}{[-3 \ \ 2 \ \ 6]^T}, \overset{v_3}{[2 \ \ -6 \ \ 3]^T} \},$$

$$\mathcal{C} = \left\{ \underset{p_1}{\tfrac{1}{2}x(x-1)}, \underset{p_2}{-(x+1)(x-1)}, \underset{p_3}{\tfrac{1}{2}(x+1)x} \right\}.$$

Comprobar que el polinomio $\underline{6+3x}$ pertenece a la imagen de T y determinar la preimagen por T del subespacio gen $\{6+3x\}$.

$$[T]^{\mathcal{C}}_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$[T]^{C}_{B}\,[v]^{B} = [T(v)]^{C}$$

raíces $\quad 0 \quad -1 \quad 1$

p_1	p_2	p_3
$0 \ 1$	$-1 \ 1$	$0 \ -1$
$p_1(-1) = 1$	$p_2(0) = 1$	$p_3(1) = 1$

$$p(x) = 6 + 3x = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$$

$$p(-1) = \alpha_1 \underbrace{p_1(-1)}_{1} + \alpha_2 \underbrace{\cancel{p_2(-1)}}_{0} + \alpha_3 \underbrace{\cancel{p_3(-1)}}_{0}$$

$$\alpha_1 = p(-1) = 3$$
$$\alpha_2 = p(0) = 6$$
$$\alpha_3 = p(1) = 9$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

General Solution: $X = \begin{pmatrix} -1+x_3 \\ 2-2x_3 \\ x_3 \end{pmatrix} = [v]^B \Rightarrow$

$$\mathcal{B} = \{ \overset{v_1}{[6 \ \ 3 \ \ 2]^T}, \overset{v_2}{[-3 \ \ 2 \ \ 6]^T}, \overset{v_3}{[2 \ \ -6 \ \ 3]^T} \},$$

$$v = (-1+x_3)v_1 + (2-2x_3)v_2 + x_3 v_3$$

$$v = (-1+x_3)\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} + (2-2x_3)\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} =$$

$$= \begin{pmatrix} -6 + 6x_3 - 6 + 6x_3 + 2x_3 \\ -3 + 3x_3 + 4 - 4x_3 - 6x_3 \\ -2 + 2x_3 + 12 - 12x_3 + 3x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -12 \\ 1 \\ 10 \end{pmatrix}}_{\text{Sol. partic}} + x_3 \overbrace{\begin{pmatrix} 14 \\ -7 \\ -7 \end{pmatrix}}^{\in Nu(T)}$$

OTRA FORMA

$$C^{E}_{C}\,[T]^{C}_{B}\,C^{B}_{E} = [T]^{E}_{E}$$

$$\begin{pmatrix} 6 & 3 & 2 \\ 3 & 2 & 6 \\ 2 & 6 & 3 \end{pmatrix}^{(-1)} = \begin{pmatrix} \frac{6}{49} & \frac{3}{49} & \frac{2}{49} \\ \frac{-3}{49} & \frac{2}{49} & \frac{6}{49} \\ \frac{2}{49} & \frac{-6}{49} & \frac{3}{49} \end{pmatrix} = C^{B}_{E}$$

$$(C^{E}_{B})^{-1} = \begin{pmatrix} 6 & -3 & 2 \\ 3 & 2 & -6 \\ 2 & 6 & 3 \end{pmatrix}^{-1}$$

$$C^{E}_{C} = \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}$$

$$p_1 = \frac{x^2 - x}{2} \qquad p_3 = \frac{x^2 + x}{2}$$

$$p_2 = -x^2 + 1$$

$$E = \{1, x, x^2\}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} \frac{6}{49} & \frac{3}{49} & \frac{2}{49} \\ \frac{-3}{49} & \frac{2}{49} & \frac{6}{49} \\ \frac{2}{49} & \frac{-6}{49} & \frac{3}{49} \end{pmatrix} = \begin{pmatrix} \frac{6}{49} & \frac{-11}{49} & \frac{23}{49} \\ \frac{3}{7} & \frac{-2}{7} & \frac{8}{7} \\ \frac{36}{49} & \frac{-17}{49} & \frac{89}{49} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{6}{49} & \frac{-11}{49} & \frac{23}{49} \\ \frac{3}{7} & \frac{-2}{7} & \frac{8}{7} \\ \frac{36}{49} & \frac{-17}{49} & \frac{89}{49} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{3}{7} & \frac{-2}{7} & \frac{8}{7} \\ \frac{15}{49} & \frac{-3}{49} & \frac{33}{49} \\ 0 & 0 & 0 \end{pmatrix}}_{[T]^{E}_{E}}$$

$$[T]^{E}_{E}\,[v]^{E} = [6+3x]^{E} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3/7\, x_1 + -2/7\, x_2 + 8/7\, x_3 = 6 \\ 15/49\, x_1 + -3/49\, x_2 + 33/49\, x_3 = 3 \\ 0\, x_1 + 0\, x_2 + 0\, x_3 = 0 \end{cases}$$

$$\underbrace{X = \begin{pmatrix} 8 - 2x_3 \\ -9 + x_3 \\ x_3 \end{pmatrix}}_{\text{una solución particular}} = \begin{pmatrix} 8 \\ -9 \\ 0 \end{pmatrix} + x_3 \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{\in Nu(T)}$$

$$Nu(T) = \text{gen}\left\{ \begin{pmatrix} 14 \\ -7 \\ -7 \end{pmatrix} \right\} = \text{gen}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

OBS: $\begin{pmatrix} 8 \\ -9 \\ 0 \end{pmatrix} = \begin{pmatrix} -12 \\ 1 \\ 10 \end{pmatrix} + \underbrace{x_3}_{x_3 = 10/7} \begin{pmatrix} 14 \\ -7 \\ -7 \end{pmatrix}$

Son equivalentes

1. Sea $p = 1 + 2x - 3x^2$ y sea $\mathcal{B}_1 = \{1+x, q, x-x^2\}$ la base de $\mathbb{R}_2[x]$ tal que $[p]^{\mathcal{B}_1} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$. Hallar una base $\mathcal{B}_2$ de $\mathbb{R}_2[x]$ tal que

$$M_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

2. Sea $T : \mathbb{R}_2[x] \to \mathbb{R}^3$ la transformación lineal definida por

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 6 & 0 \\ 6 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix},$$

donde $\mathcal{B} = \{1 - x^2, 2x + x^2, 1 + x\}$ y $\mathcal{C} = \left\{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T, \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T\right\}$ son bases de $\mathbb{R}_2[x]$ y $\mathbb{R}^3$, respectivamente. Hallar todos los $p \in \mathbb{R}_2[x]$ tales que $T(p) = \begin{bmatrix} 12 & 12 & 6 \end{bmatrix}^T$.

3. Sea $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ la proyección de $\mathbb{R}^3$ sobre el subespacio $\{x \in \mathbb{R}^3 : x_2 - 3x_3 = 0\}$ en la dirección del subespacio gen $\left\{\begin{bmatrix} 0 & 1 & -3 \end{bmatrix}^T\right\}$. Hallar la imagen por Π del triángulo de vértices $\begin{bmatrix} 1 & 3 & 1 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}^T$.

4. Se considera el espacio euclídeo $(\mathbb{R}_2[x], \langle \cdot, \cdot \rangle)$ con el producto interno definido por

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Hallar el polinomio del subespacio gen $\{1 - 2x, 3 + 6x\}$ más cercano al polinomio $6 + 6x + 6x^2$.

5. De todas las soluciones por mínimos cuadrados de la ecuación $Ax = b$, donde

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 3 & 4 & 5 \\ 3 & 4 & 5 \end{bmatrix} \quad \text{y} \quad b = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

determinar la de norma mínima.