3.10 Se considera el espacio euclídeo $(\mathbb{R}_2[x], \langle\cdot,\cdot\rangle)$ con el producto interno definido por

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Sea $p = 3x^2 - 5x + 3$.

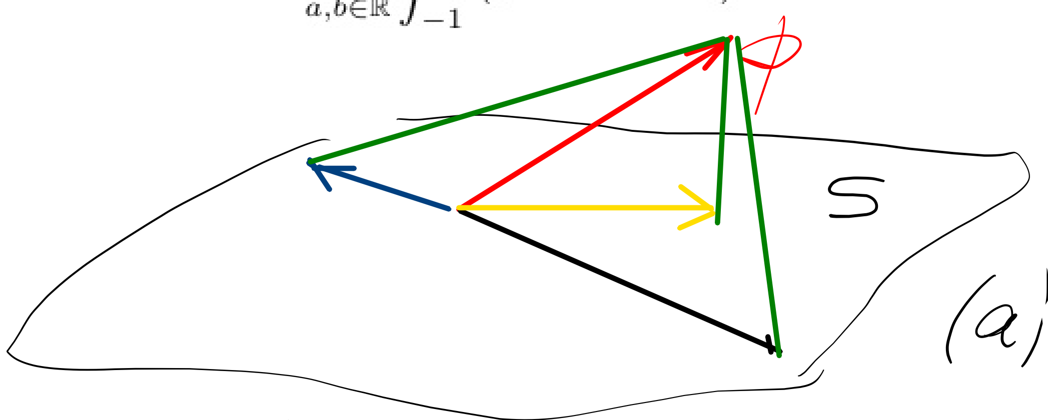
(a) Hallar el polinomio del subespacio $\mathbb{S} = \text{gen}\{3x^2 - 2x, 2x^2 + 3x\}$ más cercano al polinomio p.

(b) Calcular

$$\min_{a,b\in\mathbb{R}} \int_{-1}^{1} (p - ax^2 - bx)^2 dx.$$

$\mathbb{R}_2[x]$ esp. vectorial de dim 3

Puedo pensar en $\mathbb{R}^3$



$\dim S = 2$

$p \notin S$

(a) Me piden $P_S(p)$

$S^\perp = \text{gen}\{ax^2 + bx + c\}$ $\quad p = p_S + p_{S^\perp}$ $\quad \dim S^\perp = 1$

$\dim S + \dim S^\perp = \dim \mathbb{R}_2[x] = 3$

$$\int_{-1}^{1} (3x^2 - 2x)(ax^2 + bx + c)\,dx = \int_{-1}^{1} (3ax^4 + 3bx^3 + 3cx^2 - 2ax^3 - 2bx^2 - 2xc)\,dx$$

impares $\quad$ pares

$$= 2\int_0^1 (3ax^4 + 3cx^2 - 2bx^2)\,dx = 2\left(3a\tfrac{1}{5} + 3c\tfrac{1}{3} - 2b\tfrac{1}{3}\right) = 0$$

$\cancel{2}\left(3a\,^{1}/_{5} + 3c\,^{1}/_{3} - 2b\,^{1}/_{3}\right) = 0$

$9 \quad -10 \quad 15$

$6 \quad 15 \quad 10 \quad 9F_2 - 6F_1$

$9a + 15c - 10b = 0$ ✓

$$\int_{-1}^{1}(ax^2+bx+c)\,(2x^2+3x)\,dx = 2\int_{0}^{1}(2ax^4 + 2cx^2 + 3bx^2$$

$\cancel{2}\left(2a\,^{1}/_{5} + 2c\,^{1}/_{3} + b\right) = 0 \qquad 6a + 10c + 15b = 0$ ✓

$9 \quad -10 \quad 15$

$0 \quad 195 \quad 0 \quad b = 0$

$q \in S^{\perp} \qquad q = -5x^2 + 3$

$9a + 15c = 0 \qquad 3a + 5c$

$p = 3x^2 - 5x + 3$

$a = -5/3\,c$

$$\mathcal{P}_{S^{\perp}}\,p = \frac{\langle p, q\rangle\; q}{\|q\|^2} = \frac{\cancel{8}\,(-5x^2+3)}{\cancel{8}} = $$

$$= -5x^2 + 3$$

$$\langle p, q\rangle = \int_{-1}^{1}(3x^2-5x+3)(-5x^2+3)\,dx =$$

$$= 2\int_{0}^{1}(-15x^4 + 9x^2 - 15x^2 + 9)\,dx = 2\left(-15/5 + 9/3 - 15/3 + 9\right)$$

$$2\int_0^1 25x^4 - 30x^2 + 9\,dx = 2(5 - 10 + 9)$$

$$P_S\,p = 3x^2 - 5x + 3 - (-5x^2 + 3) = 8x^2 - 5x$$

Sea $p = 3x^2 - 5x + 3$.

$$ax^2 + bx + c \;:\; c = 0$$

(**a**) Hallar el polinomio del subespacio $\mathbb{S} = \text{gen}\{3x^2 - 2x, 2x^2 + 3x\}$ más cercano al polinomio p.

$$S = \text{gen}\{\underbrace{\overset{p_1}{x^2} ; \overset{p_2}{x}}_{BOG}\}$$

$$\int_{-1}^{1} x^2 \cdot x\,dx = 0$$

$$8x^2 - 5x$$

$$P_S\,p = \frac{\langle p\, p_1\rangle\, p_1}{\|p_1\|^2} + \frac{\langle p, p_2\rangle\, p_2}{\|p_2\|^2} = \frac{16/5}{2/5}\,x^2 - \frac{10/3}{2/5}\,x =$$

$$\int_{-1}^{1}(3x^2 - 5x + 3)\,x^2\,dx = 2\int_0^1 (3x^4 + 3x^2)\,dx = 2\left(3/5 + 1\right) = 16/5$$

$$\int_{-1}^{1}(3x^2 - 5x + 3)\,x\,dx = 2\int_0^1 -5x^2 = -10/3$$

$$\|p_1\|^2 = \int_{-1}^{1} x^4\,dx = 2\,1/5 \qquad \|p_2\|^2 = \int_{-1}^{1} x^2\,dx = 2\,1/3$$

(b) Calcular

$$\min_{a,b\in\mathbb{R}} \int_{-1}^{1} (p - ax^2 - bx)^2 \, dx.$$

$$\int_{-1}^{1} (p - ax^2 - bx)^2 dx = \| p - (\underbrace{ax^2 + bx}_{\in S}) \|^2$$

$$\langle p - ax^2 - bx ; p - ax^2 - bx \rangle$$

$$ax^2 + bx = P_S(p) = 8x^2 - 5x$$

$$\int_{-1}^{1} x^2 x \, dx = 0$$

3.11 Se considera el espacio euclídeo $(\mathbb{R}_2[x], \langle\cdot,\cdot\rangle)$ con el producto interno definido por

$$\langle a_0 + a_1x + a_2x^2, b_0 + b_1x + b_2x^2 \rangle = a_0b_0 + a_1b_1 + a_2b_2.$$

Sean $\mathbb{S} = \text{gen}\{1 - x, x - x^2\}$ y $P_{\mathbb{S}} : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ la proyección ortogonal de $\mathbb{R}_2[x]$ sobre $\mathbb{S}$. Hallar todos los $q \in \mathbb{R}_2[x]$ tales que $P_{\mathbb{S}}(q) = x - x^2$ cuya distancia a $\mathbb{S}$ es $\sqrt{3}$.

$$P_S(1-x) = 1-x$$

$$P_S(x-x^2) = x - x^2$$

$$P_S(1+x+x^2) = 0$$

$$S^\perp = \text{gen}\{p\}$$

$$p = ax^2 + bx + c \perp (1-x) \qquad \langle p (1-x) \rangle = -b + c = 0$$

$$\perp (x - x^2) \qquad \langle p (x - x^2) \rangle = -a + b = 0$$

$$a = b = c$$

$$S^\perp = \text{gen}\{1 + x + x^2\}$$

$S = \text{gen}\{x - x^2 ; 1 - x\}$ $S^{\perp} = \text{gen}\{1 + x + x^2\}$

$g = x - x^2 + \alpha(1 + x + x^2)$

$\|P_{S^{\perp}} g\|^2 = 3 = \|\alpha(1 + x + x^2)\|^2 = |\alpha|^2 \cdot 3 \Rightarrow \alpha = \pm 1$

$g = x - x^2 \pm (1 + x + x^2)$

3.14 Sea $A \in \mathbb{R}^{3\times 3}$ la matriz definida por

$C_1 \quad C_2 \quad C_1 + C_2 = C_3$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

$\{C_1\ C_2\ C_3\}$ LD

$\dim \text{Col}(A) = 2$

En cada uno de los siguientes casos, hallar las soluciones por mínimos cuadrados de la ecuación $Ax = b$, determinar la de norma mínima, y calcular el error cuadrático $\min_{x\in\mathbb{R}^3} \|b - Ax\|^2$

(a) $b = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T$.

(b) $b = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^T$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$\text{Col}(A) = \{v \in \mathbb{R}^3 \quad x + y + z = 0\}$

$\alpha = x \qquad -\beta = y$

$-\alpha + \beta = z$

$-x - y = z \rightarrow x + y + z = 0$

$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \notin \text{Col}(A) \Rightarrow$ tengo que resolver por C.M

$A^T A \hat{x} = A^T b$

$\hat{x}$ no es único

$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\hat{x} = \underbrace{x_f}_{\in \mathrm{Fil}(A)} + \underbrace{x_h}_{\in \mathrm{Nul}(A)}$$

$$\mathrm{Fil}(A) \oplus \mathrm{Nul}(A) = 0$$

$$(\mathrm{Nul}(A))^{\perp} = \mathrm{Fil}(A)$$

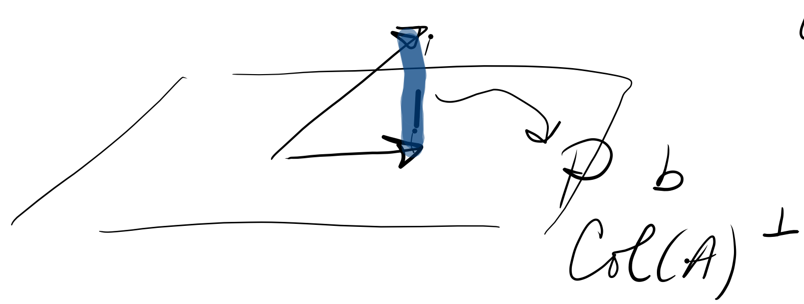
$$\hat{x} \in \mathrm{Fil}(A) \qquad \hat{x} = \begin{pmatrix} 1-\alpha \\ 2-\alpha \\ \alpha \end{pmatrix} \in \mathrm{Fil}(A) \Rightarrow$$

$$\hat{x} \perp \mathrm{Nul}(A) \Rightarrow \left\langle \hat{x} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 0 \qquad -1+\alpha-2+\alpha+\alpha = 0$$

$$\alpha = 1$$

$$\Rightarrow \hat{x}_{min} = \hat{x}_f = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \|\hat{x}_f\| = \sqrt{2}$$

Error cuadrático $\|P_{Col(A)^{\perp}} b\|^2 = \Big\| b - \underbrace{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}_{A\hat{x}} \Big\|^2 = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2 = 3$



$Ax = b$ S. Compatible

Que pase si resuelvo $A^TA\hat{x} = A^Tb$

En este caso $\hat{x} = x$

Error $= \|b - A\hat{x}\|^2 =$

$= \left\| \begin{matrix} 1/3 \\ 1/3 \\ -2/3 \end{matrix} \right\|^2 = 2/3$

$\begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}$ Incompatible

A 3×2 $\quad \in \mathbb{R}^2 \quad b \in \mathbb{R}^3$

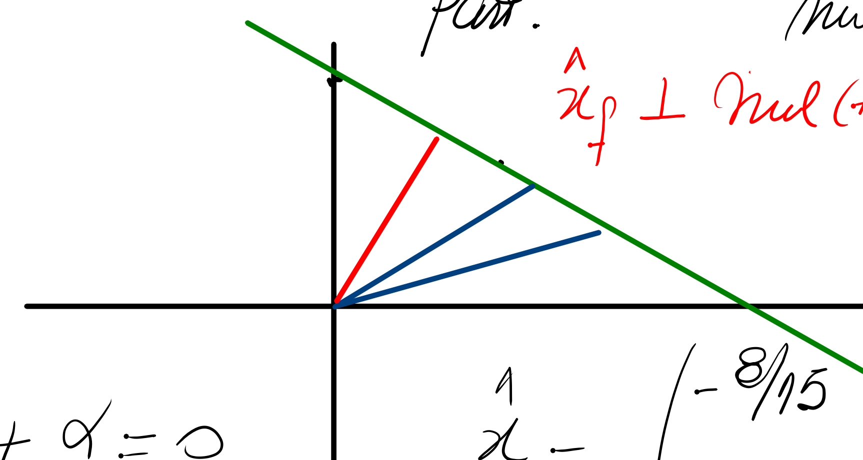
$A\hat{x} = \begin{pmatrix} 8/3 \\ 8/3 \\ 8/3 \end{pmatrix}$

$\Rightarrow$ Error $\quad A\hat{x} = P_{Col(A)}(b)$

$\hat{x} = \underbrace{\begin{pmatrix} 8/3 \\ 0 \end{pmatrix}}_{x_{Part.}} + \alpha \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{x_{nul(A)}}$

$P_{Col(A)} b = \dfrac{\left\langle \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|^2} = \dfrac{8}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\hat{x}_p \perp nul(A)$



$\left\langle \hat{x} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = 0$

$\left\langle \begin{pmatrix} 8/3 + 2\alpha \\ 0 - \alpha \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\rangle = 0$

$16/3 + 4\alpha + \alpha = 0$

$\alpha = -16/15$

$\hat{x} = \begin{pmatrix} -8/15 \\ -16/15 \end{pmatrix}$

3.16 Usando la técnica de mínimos cuadrados, ajustar los siguientes datos

x	-1	0	1	2	3
y	-14	-5	-4	1	22

mediante una recta $y = a_0 + a_1 x$, mediante una cuadrática $y = a_0 + a_1 x + a_2 x^2$, y mediante una cúbica $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. ¿Cuál de esas tres curvas se ajusta mejor a los datos?

$$y = a_0 + a_1 x$$

$$\underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \overbrace{\begin{pmatrix} -14 \\ -5 \\ -4 \\ 1 \\ 22 \end{pmatrix}}^{b}$$

$$\begin{aligned} -14 &= a_0 - a_1 \\ -5 &= a_0 \\ -4 &= a_0 + a_1 \\ 1 &= a_0 + 2a_1 \\ 22 &= a_0 + 3a_1 \end{aligned}$$

Busco $\hat{x}$ $\qquad A^T A \hat{x} = A^T b$

$$\begin{aligned} -14 &= a_0 - 1\,a_1 + 1\,a_2 \\ -5 &= a_0 + 0\,a_1 + 0\,a_2 \\ -4 &= a_0 + 1\,a_1 + 1\,a_2 \\ 1 &= a_0 + 2\,a_1 + 4\,a_2 \\ 22 &= a_0 + 3\,a_1 + 9\,a_2 \end{aligned}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = b$$

$E_{cuad} = \|b - A\hat{x}\|^2$ esta norma es mínima

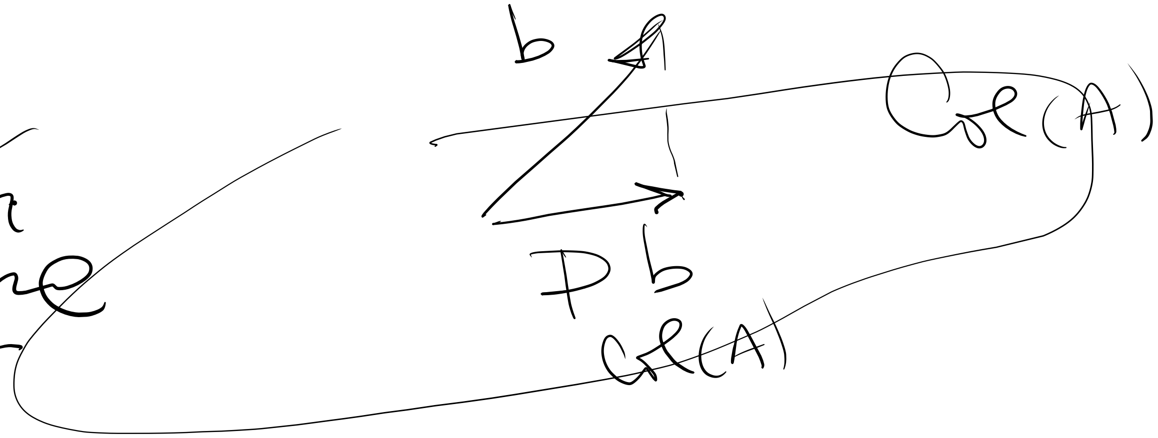
cuando $A\hat{x} = P_{Col(A)}\, b \Rightarrow b - A\hat{x} = P_{(Col(A))^{\perp}}\, b$

$\Rightarrow \|b - A\hat{x}\| = d(b, Col(A))$

$Ax = b$ Incompatible $\quad b \notin Col(A)$

$Ax \in Col(A)$

Busco que Ax sea lo más cercano a b



$\Rightarrow A\hat{x} = P_{Col(A)}(b)$

$\hat{x}$ solución que aproxime mejor

pues $A\hat{x} = P_{Col(A)}\, b$