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Regla de la cadena

Caso una variable

- $f: I \to \mathbb{R}$, diferenciable $t_0 \in I$
- $g: J \to \mathbb{R}$, diferenciable en $u_0 = f(t_0)$
- $f(I) \subset J$
- $h(t) = g \circ f(t) = g(f(t)) \quad \forall t \in I$

$\Bigg\}$ h es $\underbrace{\text{derivable}}_{\text{diferenciable}}$ en t_0 y

$$h'(t) = g'(f(t_0)) \cdot f'(t_0) = g'(u_0)\, f'(t_0)$$

$z = g(u)$ y $u = f(t) \Rightarrow z = h(t) = g(f(t))$

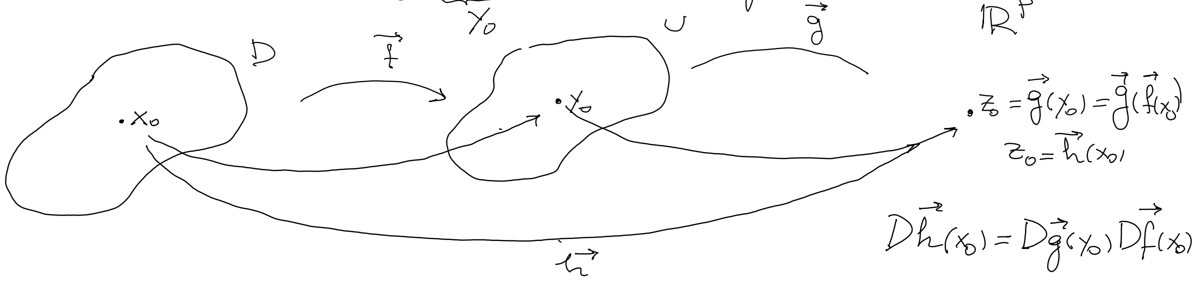
$$\frac{dz}{dt} = h'(t) = g'(u_0)\, f'(t_0) = \frac{dg}{du}(u_0)\,\frac{df}{dt}(t_0) = \frac{dz}{du}\cdot\frac{du}{dt}$$

Regla de la cadena para funciones de $\mathbb{R}^n$ a $\mathbb{R}^m$

Si $\vec{f}: D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$, es diferenciable en $x_0 \in \operatorname{int}(D)$,
$\vec{g}: U \to \mathbb{R}^p$, $U \subset \mathbb{R}^m$ y $\vec{f}(D) \subset U$, diferenciable en $y_0 = \vec{f}(x_0)$.

Entonces, $\vec{h} = \vec{g} \circ \vec{f}$ es diferenciable en x_0 y

$$D\vec{h}(x_0) = D\vec{g}(\underbrace{\vec{f}(x_0)}_{y_0})\, D\vec{f}(x_0) = D\vec{g}(y_0)\, D\vec{f}(x_0)$$



$$D\vec{h}(x_0) = D\vec{g}(y_0)\, D\vec{f}(x_0)$$

La matriz Jacobiana de la composición es el producto de las matrices jacobianas de $\vec{g}$ y $\vec{f}$.

<u>Ejemplo</u>: $\vec{g}(x,y) = (\underbrace{xy}_{g_1}, \underbrace{x+y^2}_{g_2})$, $\vec{f}(u,v) = (\underbrace{u-v}_{f_1}, \underbrace{u^2}_{f_2})$

$\vec{h} = \vec{g} \circ \vec{f}$, $\vec{h}(u,v) = \vec{g}(\vec{f}(u,v)) = (h_1(u,v), h_2(u,v))$

Hallar $D\vec{h}(1,1)$ y $\frac{\partial h_2}{\partial v}(1,1)$

$\vec{f}$ y $\vec{g}$ son diferenciables en $\mathbb{R}^2$ porque sus componentes son de clase C^1, pues son polinomios. Entonces $\vec{h}$ es diferenciable

Podemos aplicar la regla de la cadena.

$$D\vec{h}(1,1) = D\vec{g}(\vec{f}(1,1))\, D\vec{f}(1,1)$$

$$\vec{f}(1,1) = (\overbrace{u-v}^{f_1}, \overbrace{u^2}^{f_2})\Big|_{(1,1)} = (0,1)$$

$$D\vec{f}(1,1) = \begin{pmatrix} f'_{1u}(1,1) & f'_{1v}(1,1) \\ f'_{2u}(1,1) & f'_{2v}(1,1) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2u & 0 \end{pmatrix}_{(1,1)} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$$

$$\vec{g}(x,y) = (\overbrace{xy}^{g_1}, \overbrace{x+y^2}^{g_2}) \Rightarrow D\vec{g}(0,1) = \begin{pmatrix} g'_{1x}(0,1) & g'_{1y}(0,1) \\ g'_{2x}(0,1) & g'_{2y}(0,1) \end{pmatrix} = \begin{pmatrix} y & x \\ 1 & 2y \end{pmatrix}_{(0,1)}$$

$$D\vec{g}(0,1) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

$$Dh(1,1) = D\vec{g}(0,1)\, D\vec{f}(1,1) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 5 & -1 \end{pmatrix}$$

$\frac{\partial h_1}{\partial u}(1,1)$ $\frac{\partial h_1}{\partial v}(1,1)$ $\frac{\partial h_2}{\partial v}(1,1)$ $\frac{\partial h_2}{\partial u}(1,1)$

$$\boxed{\frac{\partial h_2}{\partial v}(1,1) = -1}$$

Ejemplo: Supongamos que $f(x,y)$ es una función de clase $\mathcal{C}^1(\mathbb{R}^2)$.

Sea $h(r,s) = f(\underbrace{r-s}_{x}, \underbrace{r.s}_{y})$.

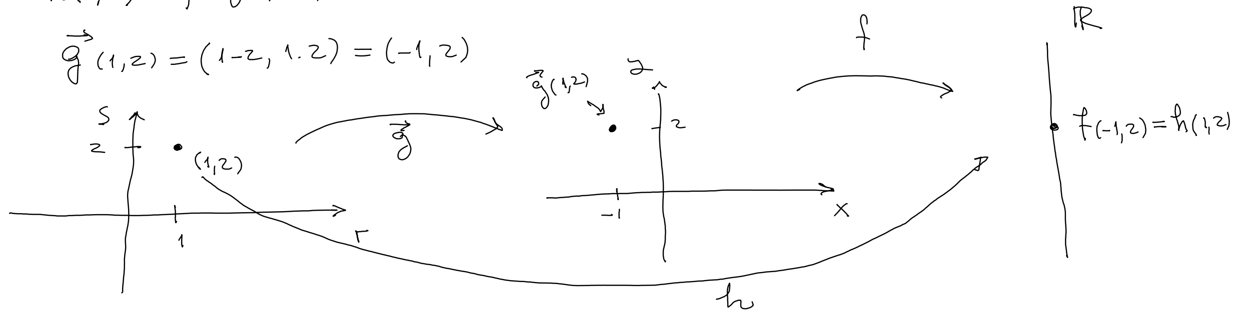
Sabiendo que

Π: $4x + 2y + 2z = 0$ es el plano tangente al gráfico de f en el punto $(-1, 2, f(-1,2))$ (plano tangente a la superficie $z = f(x,y)$), hallar el gradiente de h en $(1,2)$ y una ecuación para el plano tangente de la superficie $w = h(r,s)$ en el punto $(1, 2, h(1,2))$

Sea $\vec{g}(r,s) = (r-s, rs) = (x,y)$, $h(r,s) = f(\vec{g}(r,s)) = f \circ \vec{g}(r,s)$

$h(1,2) = f(\vec{g}(1,2)) = f(-1,2)$

$\vec{g}(1,2) = (1-2, 1.2) = (-1,2)$



Como f y $\vec{g}$ son diferenciables, f por ser $\mathcal{C}^1$ y $\vec{g}$ por tener componentes polinómicas que entonces son $\mathcal{C}^1$, entonces $h = f \circ \vec{g}$ también es diferenciable.

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Entonces, por la regla de la cadena

$$Dh(1,2) = Df(\vec{g}(1,2))\, D\vec{g}(1,2)$$

$$Dh(1,2) = Df(-1,2)\, D\vec{g}(1,2)$$

$$D\vec{g}(1,2) = \begin{pmatrix} g'_{1r}(1,2) & g'_{1s}(1,2) \\ g'_{2r}(1,2) & g'_{2s}(1,2) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ s & r \end{pmatrix}_{(1,2)} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

$\vec{g}(r,s) = (r-s, rs)$

$$Df(-1,2) = \left(\frac{\partial f}{\partial x}(-1,2) \quad \frac{\partial f}{\partial y}(-1,2) \right)$$ ¿Cómo las calculamos?

Sabemos que el plano tangente al gráfico de f en $(-1,2,f(-1,2))$ es $\Pi: \; 4x+2y+2z=0 \qquad (1)$

Sabemos que una ecuación para el plano tangente es:

$$\Pi: \quad z = f(-1,2) + \frac{\partial f}{\partial x}(-1,2)(x+1) + \frac{\partial f}{\partial y}(-1,2)(y-2)$$

Despejamos z en (1)

$$2z = -4x-2y \implies z = -2x-y = -2(x+1-1)-(y-2+2)$$

$$z = -2(x+1) + \cancel{2} - (y-2) - \cancel{2}$$

$$z = \underset{f(-1,2)}{0} + \underset{\frac{\partial f}{\partial x}(-1,2)}{(-2)}(x+1) + \underset{\frac{\partial f}{\partial y}(-1,2)}{(-1)}(y-2)$$

$f(-1,2)=0 \;\wedge\; \frac{\partial f}{\partial x}(-1,2) = -2 \;\wedge\; \frac{\partial f}{\partial y}(-1,2) = -1$

$$Df(-1,2) = \left(\frac{\partial f}{\partial x}(-1,2) \quad \frac{\partial f}{\partial y}(-1,2) \right) = (-2 \;\; -1)$$

$$Dh(1,2) = Df(-1,2)\, D\vec{g}(1,2) = (-2 \;\; -1)\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = (-4 \;\; 1)$$

$$\boxed{\nabla h(1,2) = (\underset{h'_r(1,2)}{-4}, \overset{h'_s(1,2)}{1})}$$

$$h(1,2) = f(\vec{g}(1,2)) = \boxed{f(-1,2) = 0}$$

Plano tangente a $w = h(r,s)$ en $(1,2,h(1,2)) = (1,2,0)$

Plano tangente a $\omega = h(r,s)$ en $(1,2,h(1,2)) = (1,2,0)$

Ecuación del plano tangente

$$\omega = h(1,2) + \frac{\partial h}{\partial r}(1,2)\,(r-1) + \frac{\partial h}{\partial s}(1,2)\,(s-2)$$

$$\omega = 0 + (-4)(r-1) + 1\,(s-2)$$

$$\omega = -4(r-1) + (s-2)$$

$$\omega = -4r+4+s-2 = 2-4r+s$$

$$\boxed{4r - s + \omega = 2}$$

Cálculo de derivadas parciales usando la regla de la cadena.

- $\omega = f(x,y,z)$, f diferenciable
- $(x,y,z) = (g_1(r,s), g_2(r,s), g_3(r,s)) = \vec{g}(r,s)$, $\vec{g}$ es diferenciables

$x = g_1(r,s)$, $y = g_2(r,s)$, $z = g_3(r,s)$

Si $h(r,s) = f \circ \vec{g}(r,s)$, ¿qué expresión tienen $\frac{\partial h}{\partial r}$ y $\frac{\partial h}{\partial s}$?

Por la regla de la cadena:

$$Dh(r,s) = Df(\underbrace{x,y,z}_{\vec{g}(r,s)})\, D\vec{g}(r,s)$$

$$\left[\frac{\partial h}{\partial r}(r,s) \quad \frac{\partial h}{\partial s}(r,s)\right] = \left[\frac{\partial f}{\partial x}(x,y,z) \quad \frac{\partial f}{\partial y}(x,y,z) \quad \frac{\partial f}{\partial z}(x,y,z)\right] \begin{bmatrix} \frac{\partial g_1}{\partial r}(r,s) & \frac{\partial g_1}{\partial s}(r,s) \\ \frac{\partial g_2}{\partial r}(r,s) & \frac{\partial g_2}{\partial s}(r,s) \\ \frac{\partial g_3}{\partial r}(r,s) & \frac{\partial g_3}{\partial s}(r,s) \end{bmatrix}$$

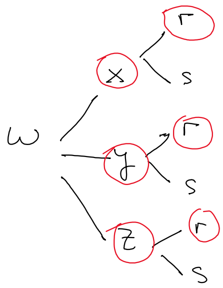
$$\frac{\partial \omega}{\partial r} = \frac{\partial h}{\partial r} = \underbrace{\frac{\partial f}{\partial x}}_{\frac{\partial \omega}{\partial x}} \cdot \underbrace{\frac{\partial g_1}{\partial r}}_{\frac{\partial x}{\partial r}} + \underbrace{\frac{\partial f}{\partial y}}_{\frac{\partial \omega}{\partial y}} \underbrace{\frac{\partial g_2}{\partial r}}_{\frac{\partial y}{\partial r}} + \underbrace{\frac{\partial f}{\partial z}}_{\frac{\partial \omega}{\partial z}} \underbrace{\frac{\partial g_3}{\partial r}}_{\frac{\partial z}{\partial r}}$$

$$\frac{\partial \omega}{\partial s} = \frac{\partial h}{\partial s} = \underbrace{\frac{\partial f}{\partial x}}_{\frac{\partial \omega}{\partial x}} \cdot \underbrace{\frac{\partial g_1}{\partial s}}_{\frac{\partial x}{\partial s}} + \underbrace{\frac{\partial f}{\partial y}}_{\frac{\partial \omega}{\partial y}} \underbrace{\frac{\partial g_2}{\partial s}}_{\frac{\partial y}{\partial s}} + \underbrace{\frac{\partial f}{\partial z}}_{\frac{\partial \omega}{\partial z}} \underbrace{\frac{\partial g_3}{\partial s}}_{\frac{\partial z}{\partial s}}$$

$$\omega = f(x,y,z) \wedge (x,y,z) = (g_1(r,s), g_2(r,s), g_3(r,s))$$

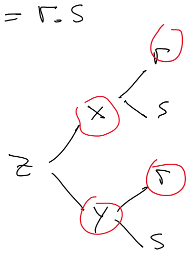
r

$w = f(x,y,z) \wedge (x,y,z) = (g_1(r,s), g_2(r,s), g_3(r,s))$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Ejemplo: $z = x^2y + y^2 \wedge x = r+s \wedge y = r.s$



$z = h(r,s)$.

Calcular $\frac{\partial h}{\partial r}(1,1)$ y $\frac{\partial h}{\partial s}(1,1)$

$r=1 \wedge s=1 \Rightarrow x=2 \wedge y=1$

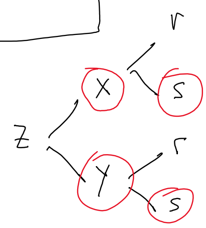
$$\frac{\partial h}{\partial r} = \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \underbrace{2xy}_{\frac{\partial z}{\partial x}}\cdot\underbrace{1}_{\frac{\partial x}{\partial r}} + \underbrace{(x^2+2y)}_{\frac{\partial z}{\partial y}}\underbrace{s}_{\partial y/\partial r}$$

$$\frac{\partial h}{\partial r} = 2xy + (x^2+2y)s \qquad x = r+s,\quad y = rs$$

$$\frac{\partial h}{\partial r}(1,1) = 2xy+(x^2+2y)s\Big|_{\substack{r=1 \; x=2\\ s=1 \; y=1}} = 2.(2).1 + (2^2+2.1).1 = 4+6 = \boxed{10}$$

$$\boxed{\frac{\partial h}{\partial r}(1,1) = 10}$$

$$\frac{\partial h}{\partial s} = \frac{\partial z}{\partial s}$$



$z = x^2y + y^2$ $\quad$ $x = r+s$, $y = r.s$ $\quad$ $\frac{\partial x}{\partial r} = 1$, $\frac{\partial x}{\partial s} = 1$, $\frac{\partial y}{\partial r} = s$, $\frac{\partial y}{\partial s} = r$

$$\frac{\partial h}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy.1 + (x^2+2y)r$$

$$\frac{\partial h}{\partial s}(1,1) = 2xy + (x^2+2y)r\Big|_{\substack{r=1 \; x=2\\ s=1 \; y=1}} = 2.2.1 + (2^2+2).1 = \boxed{10}$$

$$\boxed{\nabla h(1,1) = (10,10)}$$

Ejemplo: Sea $h(x,y) = f(x+y, x-y)$ $\quad f \in \mathcal{C}^1(\mathbb{R}^2)$

Calcular $\frac{\partial h}{\partial y}(x,y) = ?$

$z = f(r,s) \qquad r = x+y \wedge s = x-y$

$$z = f(r,s) \qquad r = x+y \ \wedge \ s = x-y$$

$$\frac{\partial h}{\partial y} = \frac{\partial z}{\partial y}$$

$$\begin{tikzcd}[row sep=small]
 & r \arrow[r] \arrow[dr] & x \\
z \arrow[ur] \arrow[dr] & & y \\
 & s \arrow[r] \arrow[dr] & x \\
 & & y
\end{tikzcd}$$

$$\frac{\partial h}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial y}$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial r}\cdot 1 + \frac{\partial f}{\partial s}\cdot(-1)$$

$$\frac{\partial h}{\partial y}(x,y) = \frac{\partial f}{\partial r}(r,s)\cdot 1 - \frac{\partial f}{\partial s}(r,s)\Bigg|_{\substack{r=x+y \\ s=x-y}} = \frac{\partial f}{\partial r}(x+y,x-y) - \frac{\partial f}{\partial s}(x+y,x-y)$$

$$\boxed{\frac{\partial h}{\partial y}(x,y) = \frac{\partial f}{\partial r}(x+y,x-y) - \frac{\partial f}{\partial s}(x+y,x-y)}$$