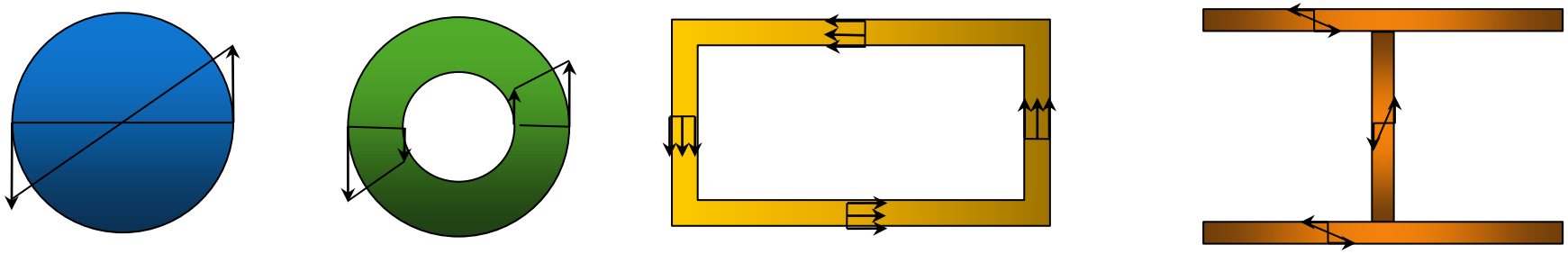
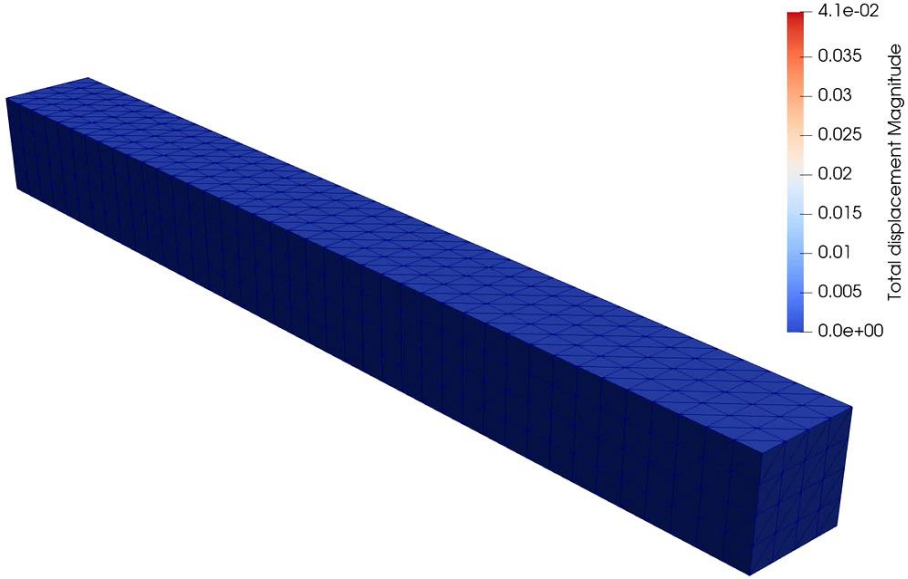
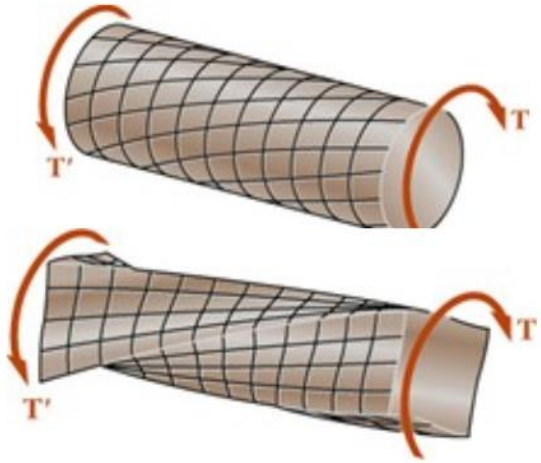
# Solicitación por Torsión en Régimen Elástico



Constanza Ruffinelli – Tania Poletilo – Manuela Medina

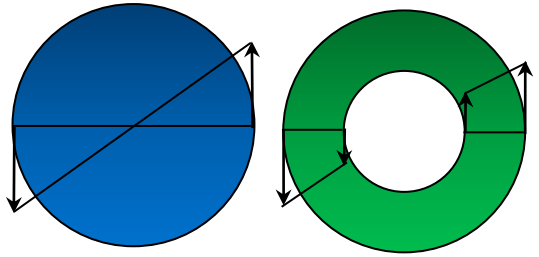
# Repaso clase teórica




## Coulomb

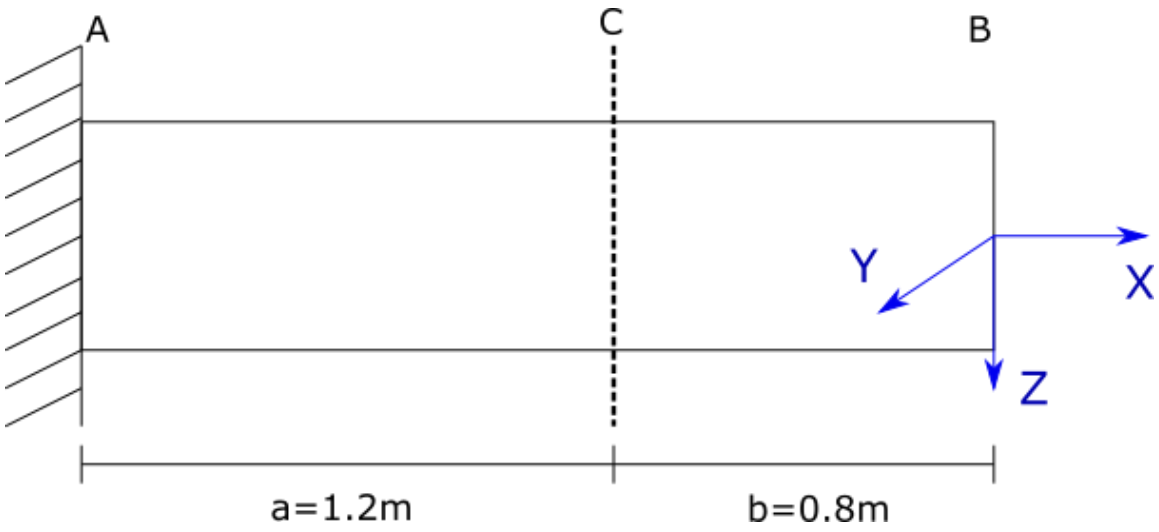
Secciones circulares: macizas o huecas

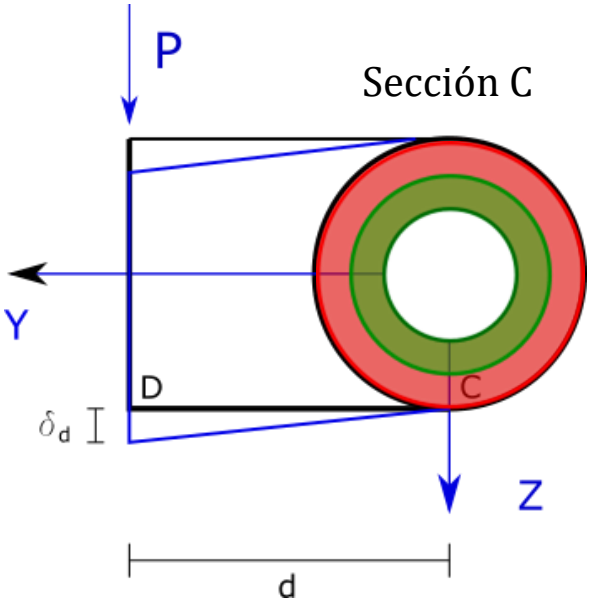


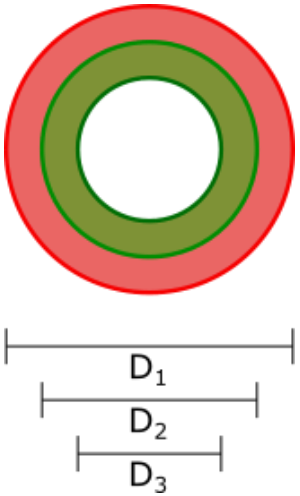
$$\tau = \frac{M_t}{J_p} \cdot r \qquad J_p^{Cir} = \frac{\pi}{32} \cdot D^4$$

$$\chi = \frac{M_t}{J_p \cdot G} \qquad J_p^{An} = \frac{\pi}{32} \cdot (D_e{}^4 - D_i{}^4)$$

# **Ejercicio 1:** Calcular la carga P que genera un desplazamiento $\delta_d$ en el punto D. Realizar diagramas de $\tau$, $M_t$, $\chi$ y $\Theta$




Datos:

$D_1 = 7{,}5\ cm$

$D_2 = 6{,}5\ cm$

$D_3 = 4{,}9\ cm$

$G_{al} = 28000\ MPa$

$\tau_{al}^{adm} = 100\ \frac{kN}{cm^2}$

$G_{br} = 36000\ MPa$

$\tau_{br}^{adm} = 60\ \frac{kN}{cm^2}$

$d = 0{,}2\ m$

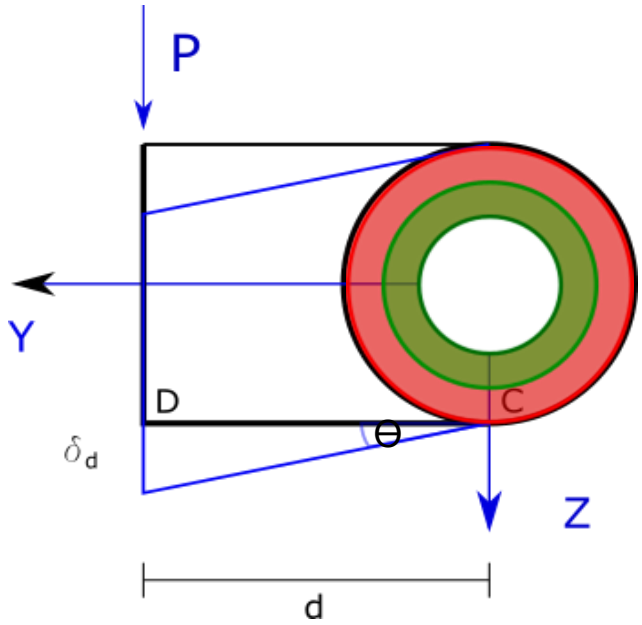
$\delta_d = 2\ cm$

## 1) Solicitaciones

La fuerza $P$ genera un $M_t$ en la barra

$$M_{tot} = P \cdot d = P \cdot 0{,}2\ m$$

$$P = \frac{M_{tot}}{0{,}2\ m}$$




## 2) Desplazamientos

Conociendo el desplazamiento del punto D puedo determinar el giro θ en la sección C de la barra

Como trabajamos con hipótesis de linealidad cinemática los desplazamientos son muy pequeños y por lo tanto consideramos $\theta = \mathrm{tg}(\theta)$

$$\tan(\theta_c) = \theta_c = \frac{\delta_d}{d} = \frac{2\ cm}{0{,}2\ m} = 0{,}1\ rad$$

Necesitamos saber cómo se reparte el momento torsor en cada uno de los materiales. Para esto necesitamos plantear una ecuación de compatibilidad de las deformaciones.

Como los materiales se deforman solidariamente, el giro en el punto C debe ser el mismo tanto para el aluminio como para el bronce.

$$\theta_c^{al} = \theta_c^{br} \qquad \chi_c^{al} = \chi_c^{br}$$

Como estamos trabajando con una sección compuesta por dos anillos utilizaremos la Teoría de Coulomb

$$\frac{Mt_{al} \cdot a}{G_{al} \cdot J_{al}} = \frac{Mt_{br} \cdot a}{G_{br} \cdot J_{br}} \qquad \frac{Mt_{al}}{G_{al} \cdot J_{al}} = \frac{Mt_{br}}{G_{br} \cdot J_{br}}$$

$$Mt_{al} = Mt_{br} \cdot \frac{G_{al} \cdot J_{al}}{G_{br} \cdot J_{br}}$$

Cálculo de Jp

$$J_{al} = \frac{\pi}{32}(D_1^4 - D_2^4) = 135{,}38\ cm^4$$

$$J_{br} = \frac{\pi}{32}(D_2^4 - D_3^4) = 118{,}65\ cm^4$$

## Ecuación de equilibrio $M_{tot} = Mt_{al} + Mt_{br}$

$$M_{tot} = Mt_{br} \cdot \frac{G_{al} \cdot J_{al}}{G_{br} \cdot J_{br}} + Mt_{br} = Mt_{br} \cdot \left( \frac{G_{al} \cdot J_{al}}{G_{br} \cdot J_{br}} + 1 \right)$$

$$0{,}53 \cdot M_{tot} = Mt_{br} \quad \Rightarrow \quad 0{,}47 \cdot M_{tot} = Mt_{al}$$

## Calculo $Mt_{br}$

$$\theta_c^{br} = \frac{Mt_{br} \cdot a}{G_{br} \cdot J_{br}} \quad \Rightarrow \quad Mt_{br} = \frac{\theta_c^{br} \cdot G_{br} \cdot J_{br}}{a}$$

con $\theta_c^{br} = \theta_c = 0{,}1\ rad$

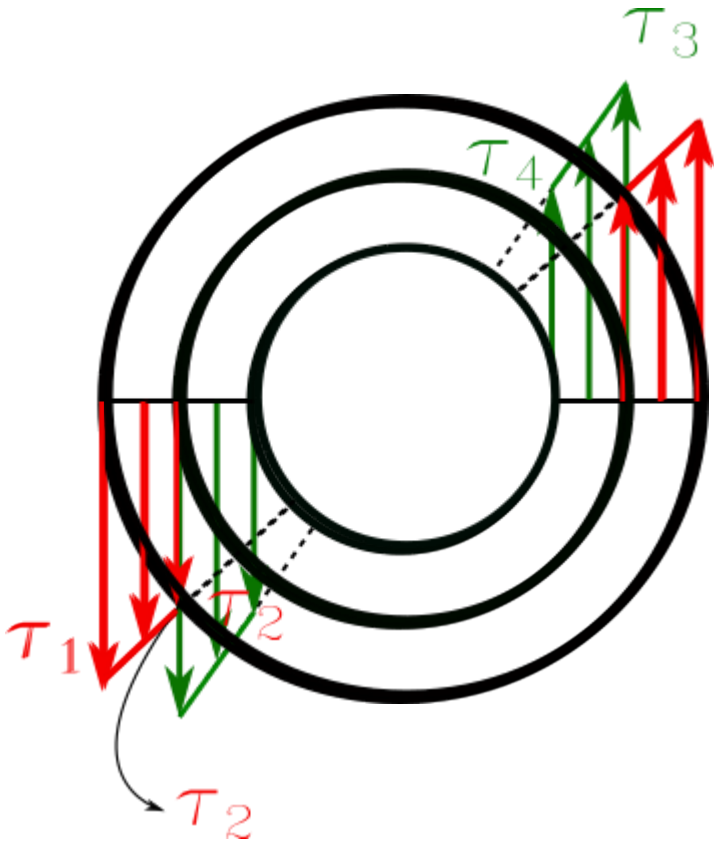
$$Mt_{br} = 355{,}96\ kN\ cm \quad \Rightarrow \quad M_{tot} = 671{,}85\ kN\ cm$$

$$Mt_{al} = 315{,}89\ kN\ cm$$

$$P = \frac{M_{tot}}{0{,}2\ m} \quad \Rightarrow \quad \boxed{P = 33{,}59\ kN}$$

# Diagrama de Tensiones Tangenciales

$$\tau = \frac{M_t}{J_P} \cdot r$$




$Mt_{al} = 315{,}89\ kN\ cm$ $\quad J_{al} = 135{,}38\ cm^4$

$$\tau_1 = \frac{Mt_{al}}{J_{al}} \cdot \frac{D_1}{2} = 8{,}74 \frac{kN}{cm^2}$$

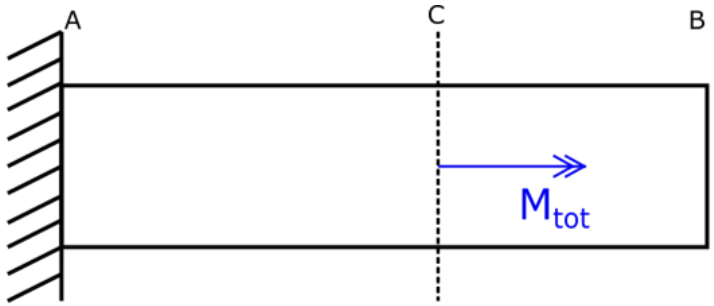
$$\tau_2 = \frac{Mt_{al}}{J_{al}} \cdot \frac{D_2}{2} = 7{,}58 \frac{kN}{cm^2}$$

$\mathrm{Mt}_{br} = 355{,}96\ kN\ cm$ $\quad J_{br} = 118{,}65\ cm^4$

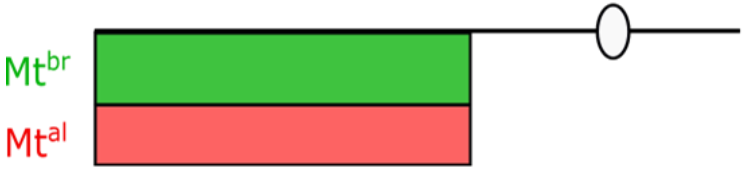
$$\tau_3 = \frac{\mathrm{Mt}_{br}}{J_{br}} \cdot \frac{D_2}{2} = 9{,}75 \frac{kN}{cm^2}$$

$$\tau_4 = \frac{\mathrm{Mt}_{br}}{J_{br}} \cdot \frac{D_3}{2} = 7{,}35 \frac{kN}{cm^2}$$

# Diagramas de $M_t$ , $\chi$ y $\theta$



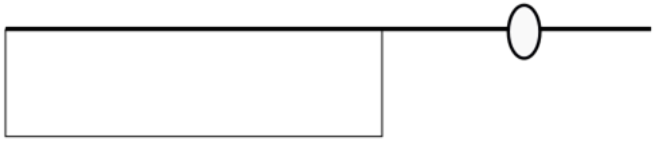

$$M_{tot} = 671{,}85\ kN\ cm$$




Mt

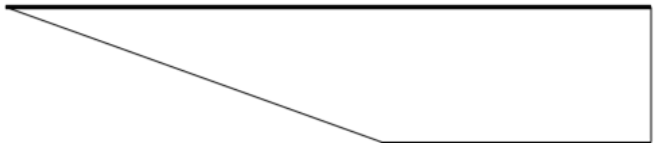
$$Mt^{br} = 355{,}96\ kN\ cm$$

$$Mt^{al} = 315{,}89\ kN\ cm$$



$\chi$

$$\chi = \frac{d\theta}{dx} = \frac{\theta_c}{a} = 8{,}33.10^{-4}\,\frac{1}{cm}$$



$\Theta$

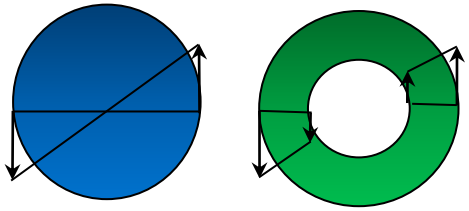
$\Theta_c$

$$\theta_c = 0.1\ rad$$

# Repaso clase teórica

## Coulomb

Secciones circulares: macizas o huecas

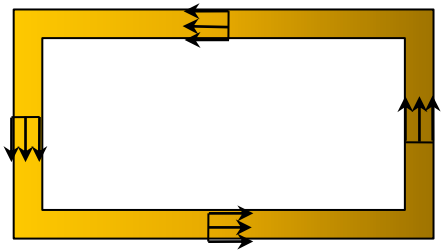


$$\tau = \frac{M_t}{J_p} \cdot r \qquad J_p^{Cir} = \frac{\pi}{32} \cdot D^4$$

$$\chi = \frac{M_t}{J_p \cdot G} \qquad J_p^{An} = \frac{\pi}{32} \cdot (D_e{}^4 - D_i{}^4)$$

## Bredt

Secciones cerradas de pequeño espesor



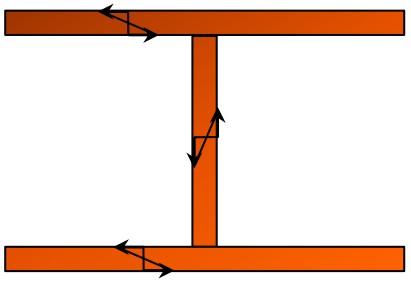
$$\tau = \frac{M_t}{2 \cdot \Omega \cdot e} \qquad J_{tb} = \frac{4 \cdot \Omega^2}{\int \frac{1}{e} dl}$$

$$\chi = \frac{Mt}{J_{tb} \cdot G}$$

Ω es el área encerrada por la línea media

## Saint Venant

Secciones abiertas de pequeño espesor

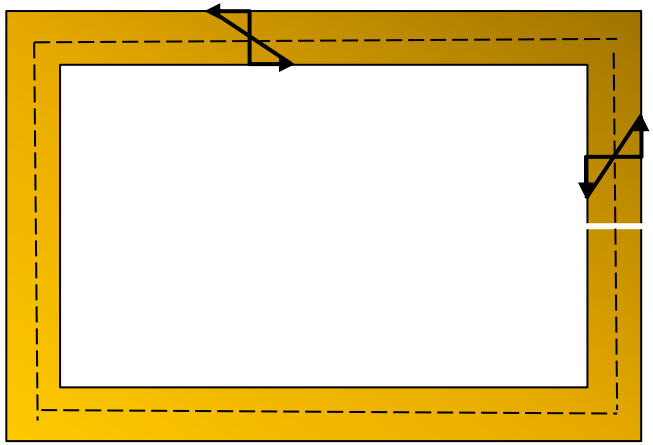


$$\tau = \frac{M_t}{J_{ts}} \cdot e \qquad J_{ts} = \frac{1}{3} \cdot \sum (a_i \cdot e_i^3)$$

$$\chi = \frac{M_t}{J_{ts} \cdot G}$$

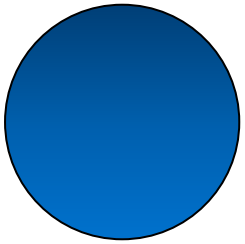
# ¡Observación!

Es importante notar que la sección debe ser completamente cerrada para poder usar Bredt. Si tuviese un corte infinitesimal en la sección tendría que usar Saint Venant

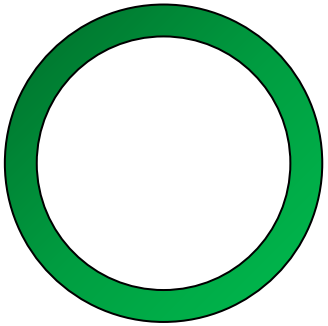


**Ejercicio 2:** Para cada una de las secciones calcular $\tau$ y $\chi$.
¿Con cuál tendrá mayores tensiones y deformaciones?
¿Por que? (Tener en cuenta que todas las áreas son iguales)

$M_t = 1\ kN\ m$ $G = 80\ GPa$ $A \cong 33{,}18\ cm^2$

1. 

$D_M = 6{,}5\ cm$

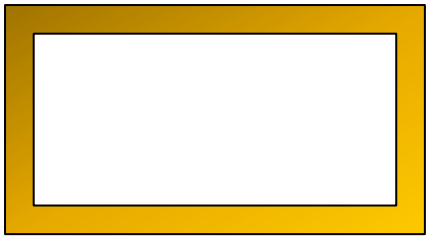
2. 

$D_e = 18{,}5\ cm$

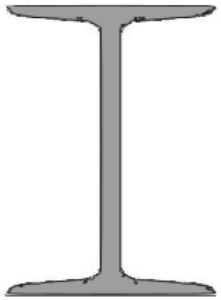
$e = 6\ mm$

$D_i = D_e - 2 \cdot e = 17{,}3\ cm$

3. Tubo (200x100x6)mm



4. IPN 200

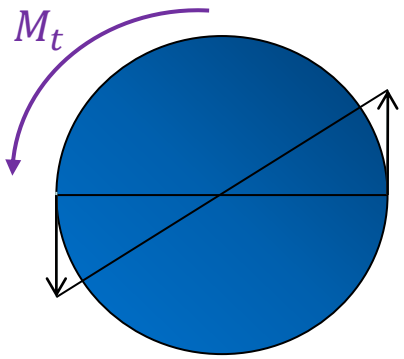


$e_{ala} = 11{,}3\ mm$ $l_{ala} = 90\ mm$

$e_{alma} = 7{,}5\ mm$ $h = 200\ mm$

# Resolución:

## 1. Circular maciza (Coulomb):

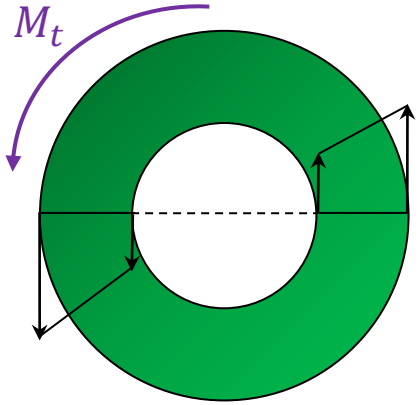



$$J_p = \frac{\pi}{32} \cdot D^4 \Rightarrow J_p = \frac{\pi}{32} \cdot (6{,}5\ cm)^4 = 175\ cm^4$$

$$\tau = \frac{M_t}{J_p} \cdot r \Rightarrow \tau = \frac{1\ kN\ m}{175\ cm^4} \cdot \frac{6{,}5\ cm}{2} = 18{,}54\ MPa$$

$$\chi = \frac{M_t}{J_p \cdot G} \Rightarrow \chi = \frac{1\ kN\ m}{175\ cm^4 \cdot 80\ GPa} = 7{,}14 \cdot 10^{-5}\ \frac{1}{cm}$$

## 2. Anular (Coulomb):

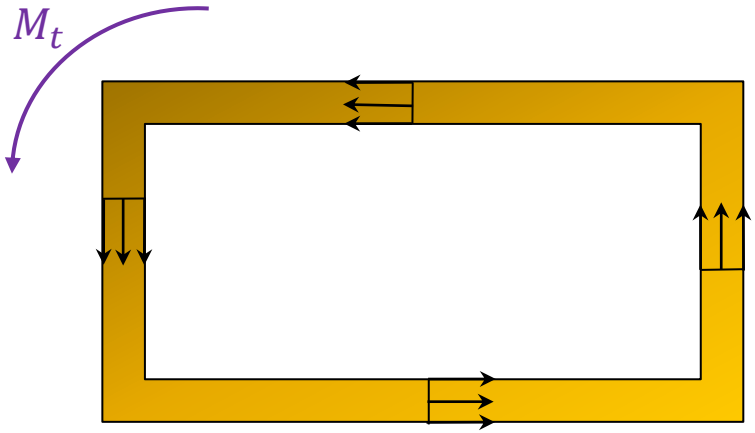



$$J_p = \frac{\pi}{32} \cdot (D_e^4 - D_i^4) \Rightarrow J_p = \frac{\pi \cdot [(18{,}5\ cm)^4 - (17{,}3\ cm)^4]}{32} = 2706\ cm^4$$

$$\tau = \frac{M_t}{J_p} \cdot r \Rightarrow \tau = \frac{1\ kN\ m}{2706\ cm^4} \cdot \frac{18{,}5\ cm}{2} = 3{,}42\ MPa$$

$$\chi = \frac{M_t}{J_p \cdot G} \Rightarrow \chi = \frac{1\ kN\ m}{2706\ cm^4 \cdot 80\ GPa} = 4{,}62 \cdot 10^{-6}\ \frac{1}{cm}$$

# 3. Tubo (Bredt)




Calculo el área Ω

**Importante**

El área Ω que se calcula es la encerrada por la línea media

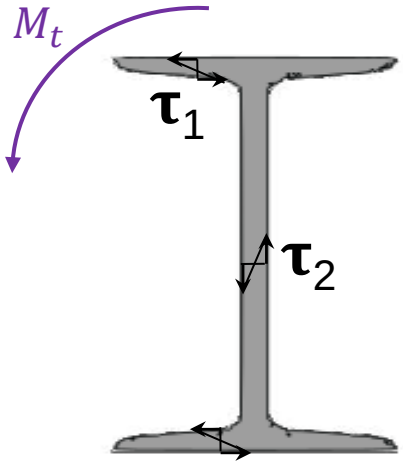
$$\Omega = (200-6)mm \cdot (100-6)mm = 182{,}36\ cm^2$$

$$J_t = \frac{4 \cdot \Omega^2}{\int \frac{1}{e}\,dl} \Rightarrow J_t = \frac{4 \cdot (182{,}36\ cm^2)^2}{\int \frac{1}{6\ mm}\,dl} = \frac{133020\ cm^4}{2 \cdot \frac{(200-6)mm}{6\ mm} + 2 \cdot \frac{(100-6)mm}{6\ mm}} = 1385\ cm^4$$

$$\tau = \frac{M_t}{2 \cdot \Omega \cdot e} \Rightarrow \tau = \frac{1\ kN\ m}{2 \cdot 182{,}36 cm^2 \cdot 0{,}6cm} = 4{,}58\ MPa$$

$$\chi = \frac{Mt}{J_t \cdot G} \Rightarrow \chi = \frac{1\ kN\ m}{1385{,}63\ cm^4 \cdot 80\ GPa} = 9{,}025 \cdot 10^{-6}\,\frac{1}{cm}$$

# 4. IPN 200 (Saint Venant)




$$J_t = \frac{1}{3} \cdot \sum(a_i \cdot e_i^3)$$

**Importante**

- Se deben sumar los términos de cada uno de los rectángulos (existe más de una forma de hacer esto)
- El valor que va elevado al cubo es el lado más chico del rectángulo

$$J_t = \frac{1}{3} \cdot [2 \cdot (90\ mm \cdot (11{,}3\ mm)^3) + (200 - 2 \cdot 11{,}3) mm \cdot (7{,}5\ mm)^3] = 11{,}15\ cm^4$$

**¡No era necesario calcularlo porque está en tabla!** $\Rightarrow$ $J_t = 11{,}2\ cm^4$

$$\tau_1 = \frac{M_t}{J_t} \cdot e_1 \Rightarrow \tau_1 = \frac{1\ kN\ m}{11{,}2\ cm^4} \cdot 11{,}3\ mm = 100{,}9\ MPa$$

$$\tau_2 = \frac{M_t}{J_t} \cdot e_2 \Rightarrow \tau_2 = \frac{1\ kN\ m}{11{,}2\ cm^4} . 7{,}5\ mm = 66{,}96\ MPa$$

$$\chi = \frac{M_t}{J_t \cdot G} \Rightarrow \chi = \frac{1\ kN\ m}{11{,}2\ cm^4 \cdot 80\ GPa} = 1{,}1 \cdot 10^{-3} \frac{1}{cm}$$

1. Circular maciza (Coulomb):

$$\tau = \frac{1\ kN\ m}{175\ cm^4} \cdot \frac{6{,}5\ cm}{2} = 18{,}54\ MPa \qquad \chi = \frac{1\ kN\ m}{175\ cm^4 \cdot 80\ GPa} = 7{,}14 \cdot 10^{-5}\ \frac{1}{cm}$$

2. Anular (Coulomb):

$$\tau = \frac{1\ kN\ m}{2706\ cm^4} \cdot \frac{18{,}5\ cm}{2} = 3{,}42\ MPa \qquad \chi = \frac{1\ kN\ m}{2706\ cm^4 \cdot 80\ GPa} = 4{,}62 \cdot 10^{-6}\ \frac{1}{cm}$$

3. Tubo (Bredt):

$$\tau = \frac{1\ kN\ m}{2 \cdot 182{,}36 cm^2 \cdot 0{,}6 cm} = 4{,}58\ MPa \qquad \chi = \frac{1\ kN\ m}{1385{,}63\ cm^4 \cdot 80\ GPa} = 9.025 \cdot 10^{-6}\ \frac{1}{cm}$$

4. IPN 200 (Saint Venant)

$$\tau = \frac{1\ kN\ m}{11{,}2\ cm^4} \cdot 11{,}3\ mm = 100{,}9\ MPa \qquad \chi = \frac{1\ kN\ m}{11{,}2\ cm^4 \cdot 80\ GPa} = 1{,}1 \cdot 10^{-3}\ \frac{1}{cm}$$

Dado que las áreas de todos los perfiles son iguales:

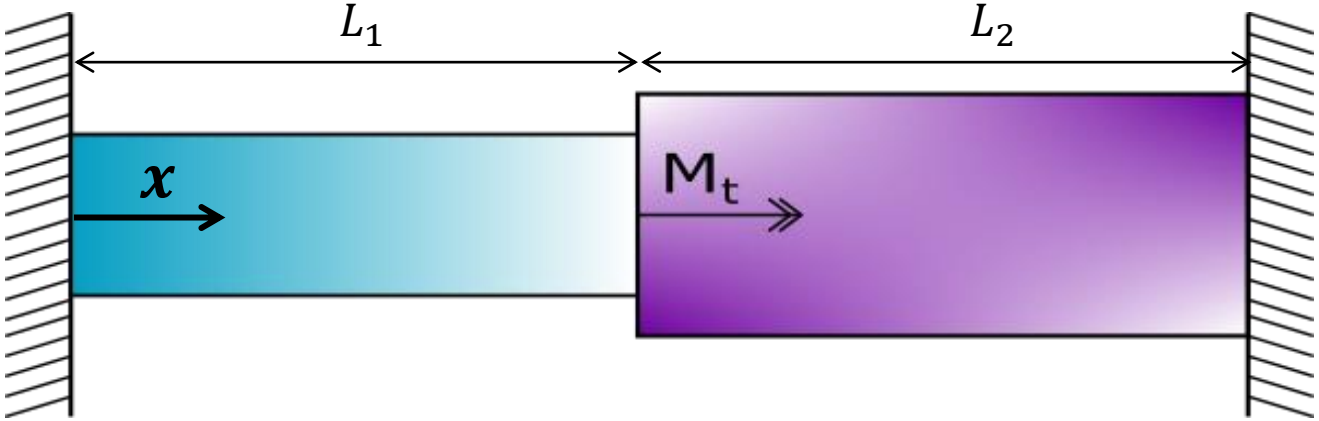
Conclusiones:

- El más conveniente es el anular, ya que el momento de inercia polar que le corresponde es mucho más grande, por lo que la curvatura y las tensiones resultantes son menores.

- El menos favorable será la sección abierta, ya que el momento de inercia a la torsión es muy chico, y por lo tanto la curvatura y tensiones resultantes serán muy grande. Es decir los perfiles abiertos no son eficientes para torsión.

- La sección maciza tampoco es eficiente, ya que el momento de inercia resulta bajo. Esto se da porque tiene material en el centro, y esto genera un menor brazo de palanca que no es eficiente para tomar torsión.

# **Ejercicio 3:** Calcular las reacciones de vínculo




Datos:

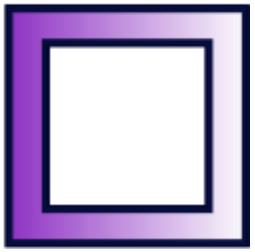
$L_1 = 2\ m$

$L_2 = 3\ m$

$M_t = 10\ kN\ m$

Sección A

Sección B

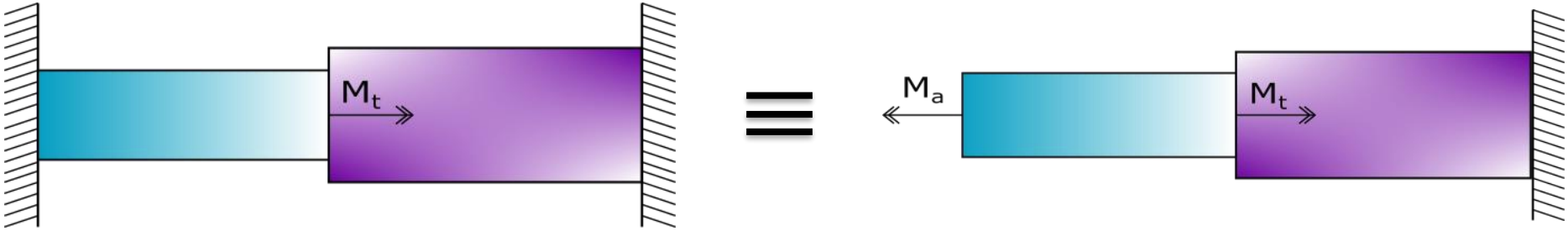


$G_a = 8000 \frac{kN}{cm^2}$ $\quad$ $G_b = 6000 \frac{kN}{cm^2}$

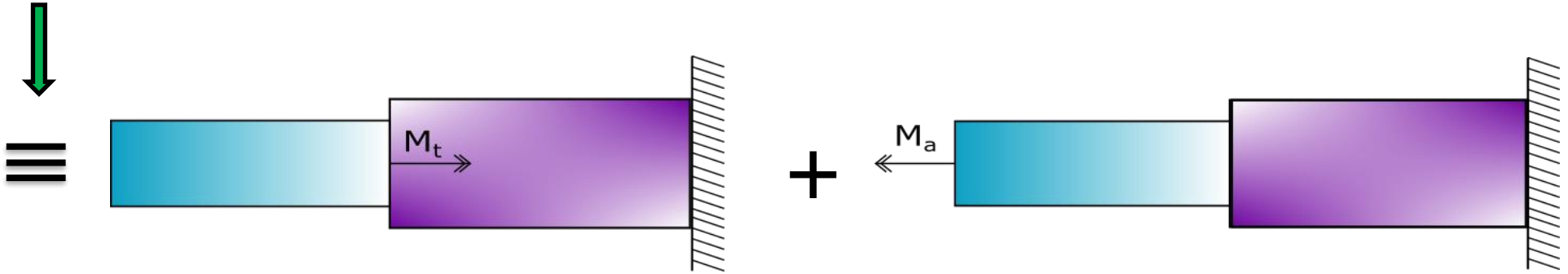
$J_{ta} = 115\ cm^4$ $\quad$ $J_{tb} = 800\ cm^4$

Para resolver éste ejercicio utilizaremos el Método de las Incógnitas Estáticas

# Método de las incógnitas estáticas




Superposición de Efectos

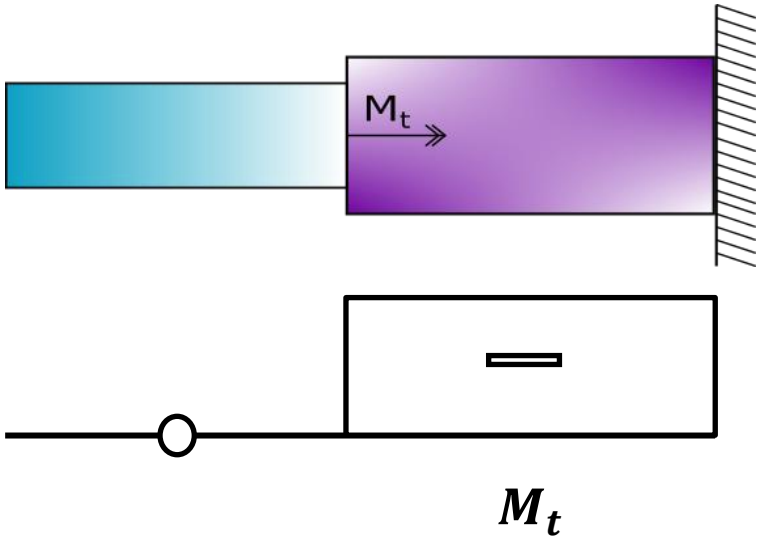



$$\theta_A^H = \theta_A^{M_A} + \theta_A^{M_t} = 0$$

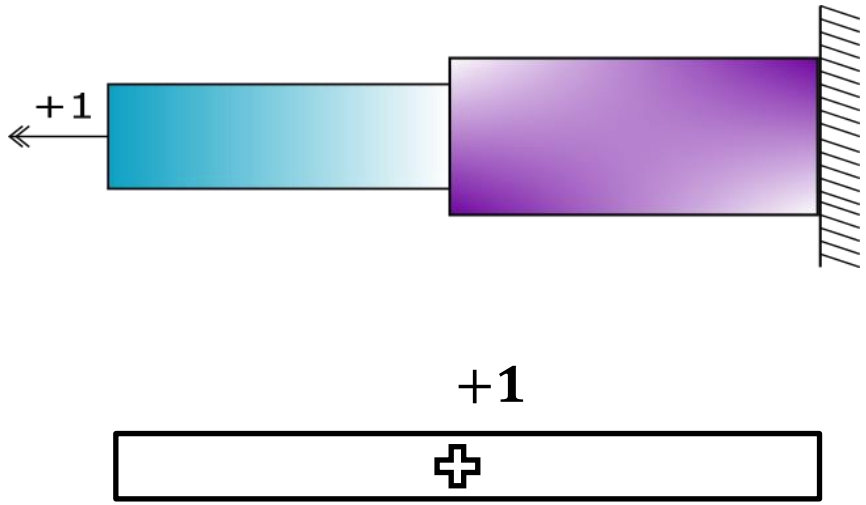
# Resolución por Teorema de los Trabajos Virtuales

Calculo el giro $\theta_A^{M_t}$

**$DV_1$**




**$SE$**

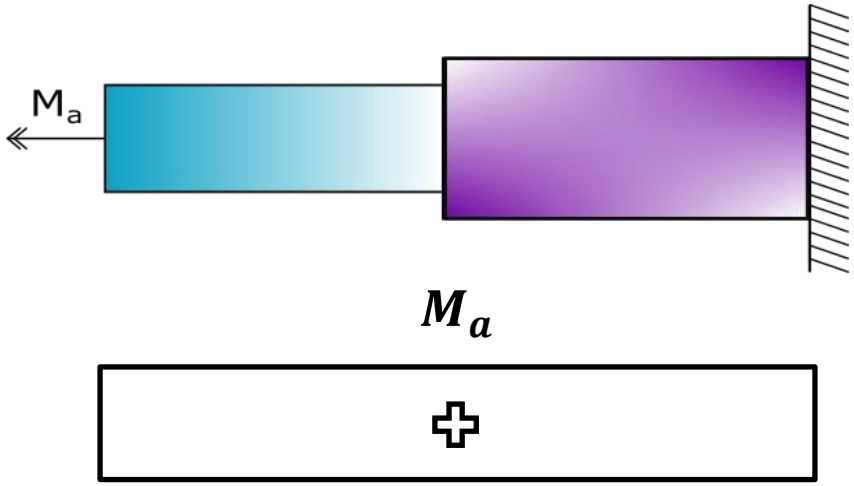



$$\theta_A^{M_t} = \int_l M_{SE}\, d\theta = \int_l M_{SE} \cdot \frac{M_{DV}}{G \cdot J}\, dx = M_{SE} \cdot \frac{M_{DV_1} \cdot L_2}{G_b \cdot J_b} = +1 \cdot \frac{-1000\ kN\ cm \cdot 300\ cm}{6000 \frac{kN}{cm^2} \cdot 800\ cm^4}$$

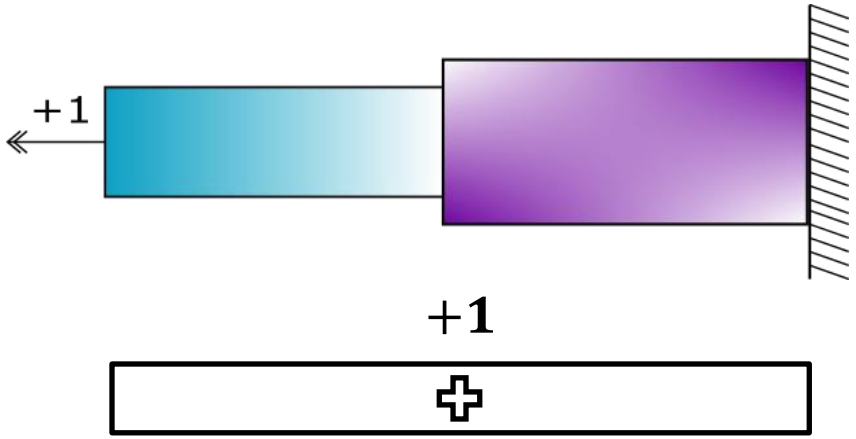
$$\theta_A^{M_t} = -0{,}0625\ rad$$

# Calculo el giro $\theta_A^{M_A}$

**$DV_2$**




**$SE$**




$$\theta_A^{M_A} = \int_l M_{SE} \cdot \frac{M_{DV}}{G \cdot J}\, dx = 1 \cdot \frac{M_a \cdot L_1}{G_a \cdot J_a} + 1 \cdot \frac{M_a \cdot L_2}{G_b \cdot J_b}$$

$$\theta_A^{M_A} = M_a \cdot \left( \frac{200\ cm}{8000\ \frac{kN}{cm^2} \cdot 115\ cm^4} + \frac{300\ cm}{6000\ \frac{kN}{cm^2} \cdot 800\ cm^4} \right)$$

$$\theta_A^{M_A} = 2{,}799 \cdot \frac{10^{-4}}{kN\ cm} \cdot M_a$$

Reemplazando en la ecuación de compatibilidad

$$\theta_A^H = \theta_A^{M_A} + \theta_A^{M_t} = 0$$

$$\theta_A^{M_A} + \theta_A^{M_t} = -0{,}0625\ rad + 2{,}799 \cdot \frac{10^{-4}\ rad}{kN\ cm} \cdot M_a = 0$$

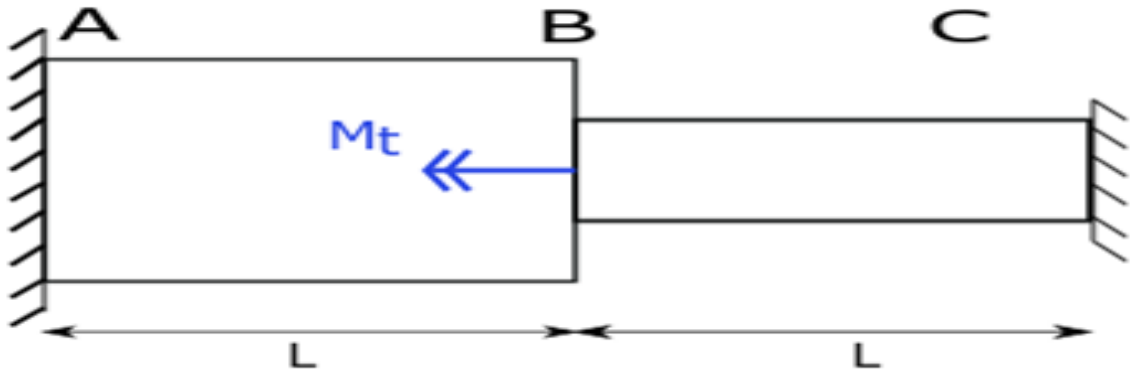
$$M_a = \frac{0{,}0625\ kN\ cm}{2{,}799\ 10^{-4}} \quad \Rightarrow \quad \boxed{M_a = 223{,}3\ kN\ cm}$$

**Aclaración**

No suele ser conveniente usar TTV para torsión, es recomendable plantearlo directamente por inspección
(o MIE pero calculando el giro directamente)

# Ejercicio 4:

1. Determinar $M_t$ admisible
2. Trazar diagramas de $M_t$, curvatura y giros
3. Tensiones tangenciales para cada sección



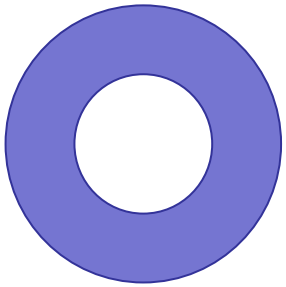

Datos:

$L = 4\ m$

$G = 85\ GPa$

$\tau_{adm} = 50\ MPa$

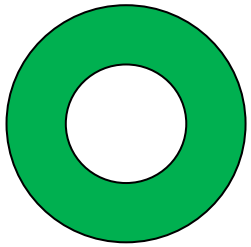
Sección AB:



$R_1 = 6\ cm$

$r_1 = 5{,}5\ cm$

Sección BC:

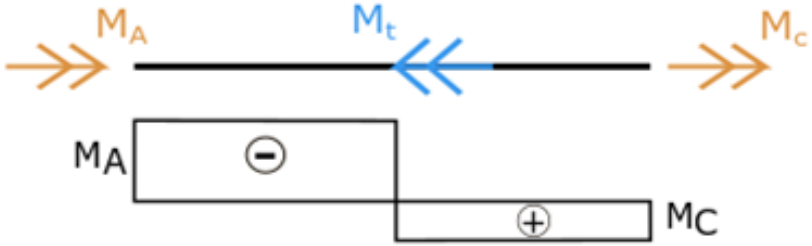


$R_2 = 4\ cm$

$r_2 = 3{,}5\ cm$

# Resolución por inspección

A) Ecuación de equilibrio $M_t = M_A + M_C$




Usamos la teoría de Coulomb

$$J_p = \frac{\pi}{32} \cdot (D_e{}^4 - D_i{}^4)$$

$$J_{AB} = 598,37\ cm^4$$

$$J_{BC} = 166,41\ cm^4$$

B) Ecuación de compatibilidad $\theta_A = 0$ $\theta_C = 0$

$$\theta_B = \theta_A + \int_A^B \chi\, dx = -\frac{M_a}{(G \cdot J)_{AB}} \cdot L_{AB}$$

$M_c = M_t - M_A$ ⟹

$$\frac{M_a}{(G \cdot J)_{AB}} \cdot L_{AB} = \frac{M_t - M_a}{(G \cdot J)_{BC}} \cdot L_{BC}$$

$$\theta_c = \theta_B + \int_B^C \chi\, dx = -\frac{M_a}{(G \cdot J)_{AB}} \cdot L_{AB} + \frac{M_c}{(G \cdot J)_{BC}} \cdot L_{BC} = 0$$

Despejando: $M_A = \dfrac{M_t \cdot J_{AB}}{J_{AB} + J_{BC}}$ ⟹

$$M_A = 0{,}782\ M_t$$

$$M_C = 0{,}218\ M_t$$

# 1. $M_{t\,adm}$: verificamos ambas secciones

## Sección 1

$$\tau_{adm} = \frac{M_A}{J_{AB}} \cdot R_1 = \frac{0{,}782 \cdot M_t}{J_{AB}} \cdot R_1$$

$$M_t = \frac{\tau_{adm} \cdot J_{AB}}{R_1 \cdot 0{,}782} = 637{,}3\ kN\ cm$$

## Sección 2

$$\tau_{adm} = \frac{M_c}{J_{BC}} \cdot R_2 = \frac{0{,}218 \cdot M_t}{J_{BC}} \cdot R_2$$

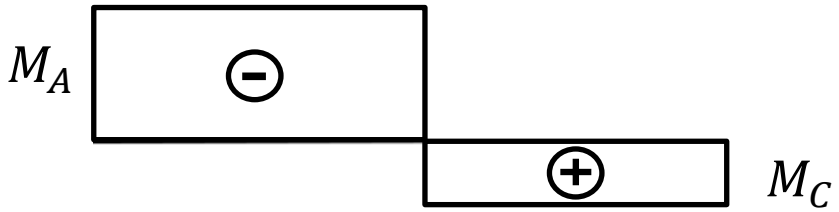
$$M_t = \frac{\tau_{adm} \cdot J_{BC}}{R_2 . 0{,}218} = 956\ kN\ cm$$

El momento máximo admisible es el menor de los dos

$$M_t = 637{,}3\ kN\ cm \Rightarrow \begin{cases} M_A = 498{,}6\ kN\ cm \\ M_C = 138{,}7\ kN\ cm \end{cases}$$

# 2. Diagramas

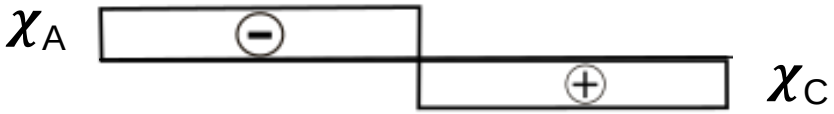
- Momento




$$M_A = 498{,}6\ kN\ cm$$

$$M_C = 138{,}7\ kN\ cm$$

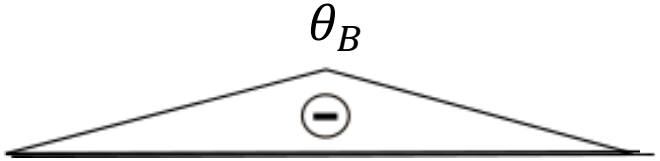
- Curvatura




$$\chi_A = \frac{M_A}{G \cdot J_{AB}} = 9{,}8 \cdot 10^{-5}\,\frac{1}{cm}$$

$$\chi_C = \frac{M_C}{G \cdot J_{BC}} = 9{,}8 \cdot 10^{-5}\,\frac{1}{cm}$$

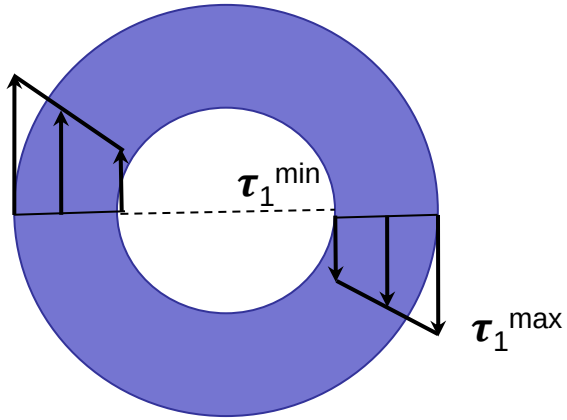
- Giro




$$\theta_B = \int \chi\, dx = \frac{M_A \cdot L}{G \cdot J_{AB}} = 0{,}039\ rad$$

# 3. Tensiones tangenciales

## Sección 1 - AB

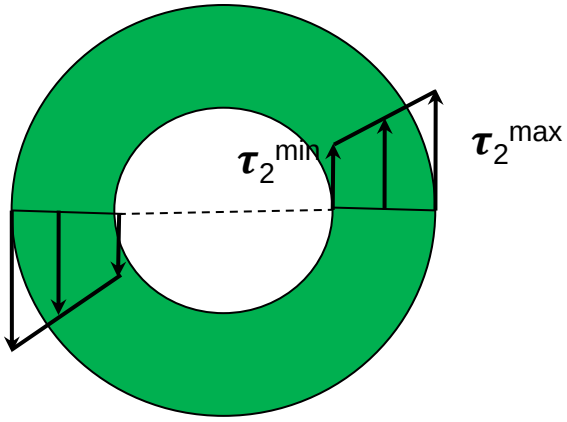



$$M_A = 498{,}6\ kN\ cm \qquad J_{AB} = 598{,}37\ cm^4$$

$$\tau_1^{mín} = \frac{M_A}{J_{AB}} \cdot r_1 = 4{,}58\,\frac{kN}{cm^2}$$

$$\tau_1^{máx} = \frac{M_A}{J_{AB}} \cdot R_1 = 5\,\frac{kN}{cm^2}$$

## Sección 2 - BC




$$M_C = 138{,}7\ kN\ cm \qquad J_{BC} = 166{,}41\ cm^4$$

$$\tau_2^{mín} = \frac{M_C}{J_{BC}} \cdot r_2 = 2{,}92\,\frac{kN}{cm^2}$$

$$\tau_2^{máx} = \frac{M_C}{J_{BC}} \cdot R_2 = 3{,}33\,\frac{kN}{cm^2}$$